REPRESENTATION THEORY AND NON-COMMUTATIVE GEOMETRY

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Contents

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1 Introduction

The theory of crossed product algebras lies at the intersection of several areas of modern mathematics: functional analysis, representation theory, topology, and noncommutative geometry. It provides a unifying framework in which one can study the relation between group symmetries and algebraic structures. In the context of this thesis, devoted to the study of periodic cyclic homology of crossed product algebras, it is natural to begin by recalling the analytic side of the story: convolution algebras, their completions into group $C^\star$-algebras, and the vast machinery developed to analyze their structure.

The starting point is classical. When a locally compact group G acts on a topological space X, the action encodes a notion of symmetry of X. It *mixes* the algebra of functions on X with the group algebra of G. The outcome is the crossed product algebra, denoted $\mathscr{C}_0(X) \rtimes G$, which contains information about both the geometry of X and the representation theory of G. Crossed products are thus the natural receptacle for invariants of group actions.

Even in the simplest case, when X is reduced to a point, the crossed product $\mathscr{C}_0(pt) \rtimes G$ is nothing other than the group $C^\star$-algebra $C^\star(G)$. This object has been studied intensively since the 1950s, following the foundational works of I. Gelfand, M-A. Naimark, and G. Segal. The universal property of $C^\star(G)$ is that it encodes all unitary representations of G, while the reduced group $C^\star$-algebra $C^\star_r(G)$ encodes all its tempered representations.

One of the most remarkable aspects of group $C^\star$-algebras is their topological rigidity. Although defined analytically as completions of the convolution algebra $\mathscr{C}_c(G)$, their K-theory groups often reflect geometric or homological invariants of G and of spaces on which G acts. For instance, when $G = \mathbb{Z}$, the Pontryagin duality shows that $C^\star(\mathbb{Z}) \simeq \mathscr{C}(S^1)$, and its K-theory is canonically identified with the topological K-theory of the circle S^1, and for compact groups G, the Peter–Weyl theorem identifies $K_0(C^\star(G))$ with the representation ring $R(G)$. In more complicated cases, such as real reductive Lie groups or non-amenable discrete groups, the computation of $K_i(C^\star_r(G))$ is a highly non-trivial problem, tied to questions in harmonic analysis, index theory, and noncommutative geometry.

The main conceptual tool to relate geometry and analysis in this setting is the *assembly map*. Roughly speaking, the assembly map takes a G-equivariant elliptic operator defined on a proper G-space and associates to it an index in the K-theory of $C^\star_r(G)$. This construction produces a homomorphism

$$\mu : K_i^G(\underline{EG}) \longrightarrow K_i(C^\star_r(G)),$$

where $\underline{EG}$ is the universal proper G-space. The Baum–Connes conjecture predicts that this assembly map is an isomorphism for all second countable locally compact groups. The conjecture lies at the heart of noncommutative geometry, with deep connections to topology (via the Novikov conjecture), to algebra (through idempotents in group rings and the Kadison–Kaplansky conjecture) and to geometry (via orbital integrals).

In the special case of reductive Lie groups, the assembly map takes a particularly elegant form. If G is a connected reductive Lie group with maximal compact subgroup K, then the homogeneous space G/K is a symmetric space of noncompact type. It carries a canonical G-invariant Spin^c-structure, and hence a Dirac operator D. For every finite-dimensional representation V of K, one may twist D by the associated homogeneous bundle, obtaining a twisted Dirac operator D_V. The Dirac induction map

$$\begin{array}{rrcc} D-\mathrm{Ind}_K^G & : R(K) & \longrightarrow & K_{\dim(G/K)}(C_r^\star(G)) \\ & [V] & \longmapsto & \mu([D_V]) \end{array}$$

is then precisely the assembly map in this context. The Connes–Kasparov theorem asserts that this map is an isomorphism, thereby computing $K_i(C_r^\star(G))$ explicitly in terms of the representation ring of K. The proof of this result is one of the landmarks of operator algebra theory: A. Wassermann proved it in 1987, using Arthur's deep structural results about the unitary dual of G, while V. Lafforgue gave a demonstration in 1998 introducing a new Banach KK-theory and provided a flexible framework encompassing both real and p-adic groups.

The remainder of this chapter is organized as follows. Section 2 recalls the theory of convolution algebras and group $C^\star$-algebras, emphasizing the role of different completions and their relation to the unitary dual. Section 3 introduces crossed product algebras, first algebraically and then analytically, with coefficients and examples. Section 4 surveys the basic facts about K-theory of $C^\star$-algebras, including Bott periodicity, Morita invariance, and illustrative examples. Section 5 presents the analytic assembly map, the Baum-Connes conjecture, the Connes–Kasparov theorem, with emphasis on Dirac induction. Section 6 discusses its connections to other related conjectures. Section 7 introduces the deformation picture via the Cartan motion group.

2 Representations and convolution algebras

The convolution algebra $\mathscr{C}_c(G)$ of a locally compact group G provides an algebraic framework for representation theory: instead of considering individual group elements, one studies linear combinations acting via convolution. This leads naturally to group $C^\star$-algebras, obtained by completing $\mathscr{C}_c(G)$ in suitable norms. The full group $C^\star(G)$ encodes all unitary representations, while the reduced group $C_r^\star(G)$ captures the tempered representations. These constructions link harmonic analysis and operator algebras. Classical examples include for instance finite groups (recovering group algebras) and abelian groups (via Fourier transform). This section introduces convolution and completions, preparing the ground for crossed products, where the group action interacts with additional algebraic data.

2.1 Group representations

We set G to be a topological group. A (complex) **representation** of G is the data of a complex Hilbert space V with a group homomorphism $\pi : G \to End(V)$ where for all $v \in V$, $g \mapsto \pi(g)(v)$ is continuous for the topology of G. We will write π, V or (π, V)

depending on the context. A **morphism of representations** between (π, V) and (τ, W) is a complex-linear map $\phi : V \to W$ such that for all $g \in G$

$$\phi \circ \pi(g) = \tau(g) \circ \phi.$$

The space of representations of G defines a category $\mathbf{Rep}(G)$ which encapsulates deep information coming from the group and is a key structure to study it. In good circumstance, it is even possible to recover the group from its representation category thanks to *Pontryagin* or *Tannaka-Krein* dualities [Bru94]. Given two representations (π, V) and (τ, W), their **direct sum** $V \oplus W$ and their **tensor product** $V \otimes W$ are representations of G for the rules:

$$(\pi \oplus \tau)(g) := \pi(g) \oplus \tau(g) \text{ and } (\pi \otimes \tau)(g) := \pi(g) \otimes \tau(g).$$

The space of *finite dimensional representations* of G is then a ring called **representation ring of** G and denoted $R(G)$. Also, we say that a representation (π, V) is **irreducible** if any closed subspace $W < V$ which is stable under $\pi(g)$ for all $g \in G$ is either $W = \emptyset$ or $W = V$.

THEOREM 2.1 *When $G = \Gamma$ is discrete, the algebra $\mathbb{C}[\Gamma]$ of complex valued Dirac functions on Γ realizes the following equivalence of categories:*

$$\mathbf{Rep}(\Gamma) \simeq \mathbb{C}[\Gamma] - \mathrm{mod}.$$

This result motivates the study of convolution algebras as they encapsulate, at least for the discrete case, the major behavior of the category of representations. It stands as an non-commutative analogy of classical geometry: whereas the study of commutative algebras provides information on the *spaces of points*, non-commutative algebras describe the *spaces of representations*.

2.2 Convolution algebra

Classical geometry associates to a space the commutative algebra of continuous functions vanishing at infinity. This algebra encapsulates all the geometrical information arising from the space. Now, for a group (discrete, compact, locally compact, p-adic, etc) we want to recover the same process: to associate a canonical algebra outlining the behavior of the group. The answer is multiple, depending on the topology of the group and the behavior we want to extract from it, but they all appear as completions of the *convolution algebra* of the group.

In this setup, the basic non-commutative algebra associated to G is the **convolution algebra** $\mathscr{C}_c(G)$ of compactly supported complex valued continuous functions on G [Bou06]. Fixing a Haar measure on G, the convolution algebra is endowed with the product:

$$(f_1 \star f_2)(g) = \int_G f_1(s) f_2(s^{-1}g) ds.$$

When $G = \Gamma$ is discrete, it reduces exactly to the group algebra $\mathbb{C}[\Gamma]$. For instance, the convolution algebra $\mathscr{C}_c(\mathbb{Z})$ is the space of complex valued $\mathbb{Z}$-sequences with Cauchy product.

The **integrated form** of a representation (π, V) of G is the map $\widehat{\pi} : \mathscr{C}_c(G) \to \mathscr{B}(V)$ from the convolution algebra to the space of bounded operators on V, defined as follows:

$$\widehat{\pi}(f)(v) := \int_G f(g)\pi(g)v\,dg. \tag{1}$$

The integrated form verifies the property that $\widehat{\pi}(f_1 \star f_2) = \widehat{\pi}(f_1) \cdot \widehat{\pi}(f_2)$ for all $f_1, f_2 \in G$. It equips the vector space V with a left-module structure over the algebra $\mathscr{C}_c(G)$ which depends only on the representation. The transformation $\pi \mapsto \widehat{\pi}$ even realizes the equivalence of categories of theorem 2.1.

When G is abelian, its **Pontryagin dual** is the space of its unitary characters:

$$\widehat{G} := \mathrm{Hom}(G, U(1)).$$

It defines a locally compact group with the property $\widehat{\widehat{G}} = G$, known as Pontryagin self-duality. The Fourier transform sends any element of the convolution algebra of G to a function vanishing at infinity on its Pontryagin dual:

$$\widehat{\bullet} : \mathscr{C}_c(G) \longrightarrow \mathscr{C}_0(\widehat{G}).$$

In the abelian case, it constructs a bridge between representation theory and classical geometry. As the convolution algebra is *commutative* if and only if the underlying group is abelian, this Fourier-type argument doesn't hold in generality. It is because the expected *dual space* or *quantum space* is not necessarily geometric but of representation type.

2.3 Completions

2.3.1 Unitary representations

We say that a representation (π, V) of G is **unitary** if for every $g \in G$ the operator $\pi(g)$ is unitary as operator on V [Mac77]. Two unitary representations (π, V) and (τ, W) are said to be **equivalent**, and we note $V \sim W$, if it exists an operator $T : V \to W$ such that for all $g \in G$, $T\pi(g) = \tau(g)T$.

We define the **unitary dual** of a locally compact group G is the set of equivalence classes of irreducible unitary representations of G:

$$\widehat{G} := \{\text{unitary irreducible representations of } G\}/\sim.$$

It is a topological space equipped with the **Fell topology** [Fel62]. When G is abelian, irreducible representations are one-dimensional and the unitary dual is nothing else than the Pontryagin dual. For non-abelian group the unitary dual is more complicated and is a central motivation for the *Langlands program* [BZSV24].

DEFINITION 2.1 *The* ***group $C^\star$-algebra*** *$C^\star(G)$ of a locally compact group G is the completion of $\mathscr{C}_c(G)$ with respect to the* ***$\star$-norm*** *defined as:*

$$\|f\|_\star := \sup_{[\pi] \in \widehat{G}} \|\widehat{\pi}(f)\|.$$

This algebra plays a key role in representation theory and non-commutative geometry, see [Dix77]. It is the $C^\star$-algebra encoding all the unitary representations of G. The Fell topology is set to obtain a bijection between the unitary dual $\widehat{G}$ and the set of primitive ideals of $C^\star(G)$. It produces a topological approach to a representation theory problem. When G is abelian, the Fourier transform $f \leftrightarrow \widehat{f}$ even gives the identification

$$C^\star(G) \simeq \mathscr{C}_0(\widehat{G}). \tag{2}$$

2.3.2 Tempered representations

A **tempered representation** is a unitary irreducible representation which is weakly contained in the left-regular representation [Fel62]. The **tempered dual** of G is defined as

$$\widehat{G}_t := \{\text{tempered representations of } G\}/\sim.$$

It is a subspace of unitary dual and inherits naturally of the induced Fell topology. Now, the corresponding $C^\star$-algebra is known as the *reduced* $C^\star$-algebra of the group.

DEFINITION 2.2 *The **reduced $C^\star$-algebra** $C_r^\star(G)$ of a locally compact group G is the completion of $\mathscr{C}_c(G)$ with respect to the **reduced $\star$-norm** defined from the integrated form of the left-regular representation λ:*

$$\|f\|_r := \|\widehat{\lambda}(f)\|.$$

When the group is compact (and more generally **amenable**), the Peter-Weyl theorem asserts that every irreducible unitary representation is tempered, which gives $C^\star(G) \simeq C_r^\star(G)$ in that case.

When G is a Lie group, the main convolution algebra is not $\mathscr{C}_c(G)$ but $\mathscr{C}_c^\infty(G)$ as we want to study **smooth** representations: the linear map $\pi : G \to GL(V)$ needs to be smooth for the topology of the Lie group. Completions of $\mathscr{C}_c^\infty(G)$ capture analytic properties of smooth representations that are difficult to study directly. The general picture links different classes of representations and different completion of the convolution algebra of the group.

2.3.3 Harish-Chandra algebra

Now, G stands as a real reductive Lie group. The **Harish-Chandra Schwartz algebra** $\mathscr{S}(G)$ of the group G is a *space of functions on G whose all derivatives decay faster than any polynomial in a certain length function* [Har66]. A FAIRE EN DEMANDANT A JULIETTE

This algebra is dense in the reduced $C^\star$-algebra $C_r^\star(G)$ and due to V. Lafforgue, N. Higson and V. Nistor, they compute the same K-theoretical invariants (see (12)). Moreover, this algebra define a great receptacle for analytic objects coming from representation theory. Indeed, due to the the Paley-Wiener theorem, which asserts that the Fourier transform identifies this algebra with certain rapidly decreasing families of operators, characters and orbital integrals extend continuously to traces on $\mathscr{S}(G)$ but not necessarily on $C_r^\star(G)$.

After the full and reduced $C^\star$-algebras, and the Harish-Chandra Schartz space, here is a condensed tabular for key examples.

Group type	Type of irreducible representations	Crossed product algebra
Locally compact	Unitary ($\widehat{G}$)	$C^\star(G)$
	Tempered ($\widehat{G}_t$)	$C_r^\star(G)$
Compact	Unitary or Tempered ($\widehat{G} = \widehat{G}_t$)	$C^\star(G) \simeq C_r^\star(G)$
Real reductive	Smooth tempered	$\mathscr{S}(G)$
p-adic	Smooth admissible finite length	The Hecke algebra $\mathscr{H}(G)$

To study these algebras some non-commutative algebraic technics are required, as cyclic homology groups for spectral invariants and K-groups for index theory and classification problems. FAIRE LE LIEN AVEC LE CHAPITRE 2

3 Crossed product algebras

Crossed product algebras extend the idea of group $C^\star$-algebras by allowing a group G to act on another algebra A. The construction $A \rtimes G$ encodes both the algebraic structure of A and the dynamics of the group action, generalizing the case $A = \mathbb{C}$ which corresponds to $C^\star(G)$. Concretely, elements are functions from G to A with convolution twisted by the action. Crossed products thus provide a noncommutative analogue of transformation group spaces and dynamical systems. They are central in noncommutative geometry: when $A = \mathscr{C}_0(X)$, the crossed product reflects the G-action on the topological space X. This section introduces algebraic crossed products, explains their functorial properties, and presents fundamental results such as Green's imprimitivity theorem. This part is deeply inspired from the great book [Wil07], which stands as the classical reference for the subject.

3.1 Algebraic crossed product

A **$C^\star$-dynamical system** is a triple (A, G, α) of a $C^\star$-algebra A, a locally compact group G and an action α of G on A by continuous automorphisms. Let $\mathscr{C}_c(G, A)$ be the space of compactly supported continuous functions. We define the twisted convolution by:

$$(f_1 \star f_2)(g) := \int_G f_1(s)\alpha_s(f_2(s^{-1}g))ds.$$

The algebra $(\mathscr{C}_c(G, A), \star)$ with this convolution product is called **algebraic crossed product** and denoted $A \rtimes G$. When $A = \mathbb{C}$ is equipped with the trivial action, we recover the usual convolution algebra $\mathscr{C}_c(G)$ of the group.

3.2 Full and reduced crossed product

A **covariant representation** of the dynamical system (A, G, α) on a Hilbert space H is the data of $\pi : A \to B(H)$, a non-degenerate representation of A and $U : G \to U(H)$ a strongly continuous unitary representation of G which intertwine as:

$$U_g \pi(a) U_g^\star = \pi(\alpha_g(a)).$$

In this context, we call **integrated form** of a covariant representation (π, U) the operator $\pi \rtimes U : \mathscr{C}_c(G,A) \to B(H)$ defined as

$$(\pi \rtimes U)(f)(v) := \int_G \pi(f(g)) U_g(v) dg.$$

It is a generalization of the integration form (1) we defined for usual convolution algebras. As before, it builds a left-module over the algebra $\mathscr{C}_c(G,A)$ which depends only on the covariant representation (π, U). The **full crossed product** $A \rtimes_\alpha G$ is the completion of $\mathscr{C}_c(G,A)$ for the norm

$$\|f\|_{\max} := \sup_{(\pi,U) \text{ covariant}} \|(\pi \rtimes U)(f)\|.$$

It satisfies the universal property that each covariant representation (π, U) integrates to a nondegenerate representation $\pi \rtimes U$ of $A \rtimes_\alpha G$. Modulo unitary equivalences, it realizes the following equivalence of categories:

$$\begin{array}{ccc} \mathrm{Cov}\mathbf{Rep}(A,G,\alpha) & \longleftrightarrow & \text{Non-Deg}\mathbf{Rep}(A \rtimes_\alpha G)/\sim \\ (\pi, U) & \longrightarrow & \pi \rtimes U \end{array}$$

This crossed product $A \rtimes_\alpha G$ encodes both the algebraic structure of A and the dynamics of the G-action. It is best thought of as the *noncommutative quotient* of the system (A,G,α). when $A = \mathbb{C}$ is endowed with the trivial action, $\mathbb{C} \rtimes_{id} G$ is nothing other than the group $C^\star$-algebra $C^\star(G)$.

The **reduced crossed product** $A \rtimes_{\alpha,r} G$ is the completion of $\mathscr{C}_c(G,A)$ for the reduced norm

$$\|f\|_r := \|(\pi_{\text{reg}} \rtimes \lambda)(f)\|.$$

where $(\pi_{\text{reg}}, \lambda)$ is the regular covariant representation on the Hilbert space $H = \mathscr{L}^2(G)$ defined as:

$$(\pi_{\text{reg}}(a))(f)(x) := \alpha_g^{-1}(a) f(g) \text{ and } (\lambda(g))(f)(x) = f(g^{-1}x).$$

When $A = \mathbb{C}$ with trivial action we recover the reduced $C^\star$-algebra $C_r^\star(G)$ of the group. Also, if G is compact (and more generally amenable), the Peter-Weyl theorem asserts that $A \rtimes_\alpha G \simeq A \rtimes_{\alpha,r} G$.

3.3 Properties of crossed product

The main properties of crossed product algebras are categorical and require the notion of Morita equivalences.Two rings A and B are **Morita equivalent** if their category of modules A-mod and B-mod are equivalent; we will write $A \sim B$. For instance, any ring A is Morita equivalent with all of its matrices spaces $\mathscr{M}_n(A)$. When A and B are $C^\star$-algebras, we forget the algebraic and $\star$-structures to check Morita equivalences.

Crossed products enjoy remarkable structural features. Green's imprimitivity theorem [Gre80] shows that they preserve Morita equivalences under restriction to subgroups and Takai duality [Rae88] stands as a Fourier inversion theorem in the operator-algebraic setting. To state these results we need the definition of Morita equivalence.

If X is topological space endowed with an action α of G, we can extend the action on $A = \mathscr{C}_0(X)$ by the formula $(\alpha_g f)(x) := f(\alpha_g^{-1}x)$. Then $(\mathscr{C}_0(X), G, \alpha)$ becomes a $C^\star$-dynamical system and defines the crossed product $\mathscr{C}_0(X) \rtimes_\alpha G$. This $C^\star$-algebra encodes the orbit structure of the action as states the following theorem from [Com84] and [Gre77].

THEOREM 3.1 *When the action of G on X is free and proper, we have the Morita equivalence:*

$$\mathscr{C}_0(X) \rtimes_\alpha G \sim \mathscr{C}_0(X/G).$$

For instance, let us take $G = \mathbb{Z}$ and $X = \mathbb{R}$, for the integer translations τ. The action of $\mathbb{Z}$ extends to $\mathscr{C}_0(\mathbb{R})$ with the formula $(\tau_n f)(x) = f(x-n)$. Under Fourier transform, the action becomes

$$\widehat{(\tau_n f)}(x) = \exp(2i\pi nx)\hat{f}(x).$$

Since $\exp(2i\pi nx) = \exp(2i\pi mx)$ for any integers, the value of $\hat{f}(x)$ depends only x mod 1 and the Fourier transform $\hat{f}$ defines a function on S^1. This approach sticks with the Morita equivalence between $\mathscr{C}_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$ and $\mathscr{C}(S^1)$.

If H is a subgroup of G, the $C^\star$-algebra $\mathscr{C}_0(G,A)$ is naturally equipped with the diagonal action of the subgroup H. The fixed points with respect to this action $\mathscr{C}_0(G,A)^H$ stands a non-commutative analogue of the construction of a vector bundle $G \times_H V$ over G/H from any representation V over H.

THEOREM 3.2 (*Green Imprimitivity*) *If G acts on A with an action α and H is a subgroup of G, there is Morita equivalence:*

$$\mathscr{C}_0(G,A)^H \rtimes_\alpha G \sim A \rtimes_{\alpha_{|H}} H.$$

This fundamental theorem tells us that the study of the crossed product algebra by a subgroup relies heavily on the understanding of the homogeneous space. Without coefficients, it stands as in [Rae92]:

$$\mathscr{C}_0(G/H) \rtimes_{\mathrm{tr.}} G \sim C^\star(H). \tag{3}$$

It provides a profound duality between inducing and restricting representations in the context of group actions and algebras. Even though the crossed product may seem far more complicated, its representation theory and structure are fundamentally governed by the subgroup H. It is one of the motivation for the Connes-Kasparov theorem, following the ideas behind Mackey's theory of induced representations and serves as a noncommutative geometric version of those classical results.

When G is abelian, the crossed product algebra $A \rtimes_\alpha G$ joys a Pontryagin self-duality property, known as Takai duality [Tak75].

THEOREM 3.3 (*Takai duality*) *If G is a locally compact abelian group which acts by α on A, then we have the Morita equivalence:*

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \sim A.$$

4 K-theory

K-theory was born at the crossroads of topology and representation theory. Its origins can be traced back to the study of vector bundles, where the need arose to compare and combine bundles in a systematic way. By taking formal differences, A. Grothendieck introduced the notion of a Grothendieck group, leading to topological K-theory due to the work of M. Atiyah and F. Hirzebruch [AH59]. This construction revealed hidden periodicities, most famously Bott periodicity, which endows the theory with a rich structure and computational power. The main early notes references are at least [Ati19], [Bot69] and [Kar08].

It provides a natural framework to organize and compare representations of groups and algebras. For instance, the algebraic K_0 of the group algebra $\mathbb{C}[\Gamma]$ recovers exactly $R(\Gamma)$ when Γ is finite. For a compact group G, one finds $K_0(C_r^\star(G)) \simeq R(G)$, showing that K-theory encodes the decomposition of unitary representations. The power of the theory becomes apparent for noncompact or noncommutative settings, where representation categories are too vast or continuous to handle directly. K-theory distills this complexity into computable invariants: classes of projections and unitaries capture *stable* aspects of representations. This is precisely what makes it central in the Baum–Connes conjecture, where $K_i(C_r^\star(G))$ serves as the receptacle for the *universal* index of equivariant elliptic operators.

K-theory unifies several perspectives: it arises from vector bundles in topology, from representation rings in algebra, and from operator algebras in analysis. This convergence is one of its deepest strengths. Far from being an abstract construction, K-theory extracts computable invariants that bridge geometry, topology, and representation theory, providing a conceptual language in which index theorems and assembly maps naturally live. Great introduction are given by [Mur90], [RLL00], [HR01] and [WO93].

4.1 The K-groups

4.1.1 For spaces

We take X to be a topological space. Let us consider the space $\mathrm{VB}(X)$ generated by the isomorphism classes of complex vector bundles over X. The **even K-theory of** X is the Grothendieck group of $\mathrm{VB}(X)$ for the direct sum of vector bundles:

$$K^0(X) := \mathrm{Gr}(\mathrm{VB}(X)).$$

It corresponds to a quotient of $\mathrm{VB}(X)$ with respect to the relation $[E] \sim [E'] + [E'']$ if we have an exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles over X. The group $K^0(X)$

is abelian and generated by the formal differences $[E]-[E'] \in \mathrm{K}_0(X)$ of such vector bundles. It defines a contravariant functor from the category of topological spaces to the category of abelian groups. This functor is homotopy invariant, which is that if X and Y are homotopic as topological spaces, then $K^0(X) \simeq K^0(Y)$.

The easiest example of computation is when $X = pt$. A vector bundle over a point is just a finite dimensional vector space. Modulo isomorphisms, finite dimensional vector spaces are classified by their dimensions and we have $K^0(pt) := \mathrm{Gr}(\mathbb{N}) \simeq \mathbb{Z}$.

The **suspension** ΣX over X is the topological space:

$$\Sigma X := (X \times [0,1])/\approx \text{, where}(x,1) \approx (y,1) \text{ and } (x,0) \approx (y,0) \text{ for all } x,y \in X.$$

The **higher K-groups of** X are defined to be:

$$K^i(X) := K^0(\Sigma^i X),$$

where $\Sigma^i X$ stands as the i-th suspension of X. As for K^0, the higher K-groups are homotopy invariant contravariant functors from the category of topological space to the category of abelian groups.

The most striking theorem is known as the **Bott periodicity theorem** which relates K-groups by parity via a natural isomorphism and makes the K-theory a $\mathbb{Z}/2\mathbb{Z}$-graded theory [Bot70].

THEOREM 4.1 (*Bott periodicity*) *For every topological space X: $K^i(X) \simeq K^{i+2}(X)$.*

Due to this theorem, we will mainly be interested in $K^0(X)$ and $K^1(X)$.

4.1.2 For algebras

We fix A to be a unital $C^\star$-algebra. We call $\mathscr{P}(A)$ the **space of idempotent matrices** or **projections** with coefficients in A. This space stands as a non-commutative analogue of the vector bundles over topological spaces. Indeed, if $A = \mathscr{C}_0(X)$ is the space of vanishing at infinity functions over a given topological space, the Serre-Swan theorem [Swa62] asserts that its space of idempotent matrices is in bijection with the space of vector bundles on X:

$$VB(X) \longleftrightarrow \mathscr{P}(\mathscr{C}_0(X)),$$

where we identify a vector bundle with its sections vanishing at infinity. The space of idempotent matrices, or projections, defines a semi-group for the diagonal block law.

We define the **K-theory of** A to be the Grothendieck group of isomorphism classes of projections on A:

$$\mathrm{K}_0(A) := \mathrm{Gr}(\mathscr{P}(A)).$$

We define the **loop algebra** ΩA to be the space of loops of source 0 in A:

$$\Omega A := \{f : [0,1] \to A \mid f \text{ continuous and } f(0) = f(1) = 0 \in A\}.$$

The **higher K-groups of** A are defined to be the K_0-groups of iterated suspension algebras:

$$\mathrm{K}_i(A) := \mathrm{K}_0(\Omega^i A).$$

The Bott periodicity theorem is also true in this algebraic context.

THEOREM 4.2 (*Bott periodicity*) *For all $C^\star$-algebra A: $\mathrm{K}_i(A) \simeq \mathrm{K}_{i+2}(A)$.*

Thus we will be mainly interested in $\mathrm{K}_0(A) = \mathrm{Gr}(\mathscr{P}(A))$ and $\mathrm{K}_1(A) = \mathrm{K}_0(\Omega A)$. Despite the definition of $\mathrm{K}_1(A)$ in terms of idempotent matrices over the loop algebra, we have another equivalent description. Two unitary matrices of any size $u, v \in U_\infty(A)$ are said to be homotopic, and we note $u \sim v$, if there exists a unitary path connecting them. We can compare matrices of difference sizes as we embed iteratively $U_n(A) \subseteq U_{n+1}(A)$ on the up-left corner. We can describe $\mathrm{K}_1(A)$ as the space of unitary matrices of any dimension over A modulo homotopy:

$$\mathrm{K}_1(A) \simeq U_\infty(A)/\sim. \tag{4}$$

We recall that the Serre-Swan theorem states that the space of projections over $\mathscr{C}_0(X)$ is in bijection with vector bundles over X. Also, the loop algebra $\Omega\mathscr{C}_0(X)$ coincides with $\mathscr{C}_0(\Sigma X)$ as suggested by the Eckmann–Hilton duality [Fuk66], which induces the following theorem.

THEOREM 4.3 *For all topological space X, the following are isomorphisms of abelian groups:*

$$K_0(\mathscr{C}_0(X)) \simeq K^0(X) \text{ and } K_1(\mathscr{C}_0(X)) \simeq K^1(X).$$

4.2 Properties of K-groups

The groups K_0 and K_1 define functors from the category of $C^\star$-algebras to the category of abelian groups which preserve *algebraic* homotopy invariances. We say that two $\star$-homomorphisms $\phi, \psi : A \to B$ are **homotopic** if there exists a path of $\star$-homomorphisms $\phi_t : A \to B$ connecting ϕ and ψ and such that $t \mapsto \phi_t(a)$ is continuous for all $a \in A$. Two $C^\star$-algebras A and B are said to be **homotopic** if it exists $\star$-homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \phi$ is homotopic to id_A and $\phi \circ \psi$ is homotopic to id_B. As a direct consequence of their definition, the groups K_0 and K_1 respect these properties.

PROPOSITION 4.4 *If A and B are two homotopic $C^\star$-algebras:*

$$\mathrm{K}_0(A) \simeq \mathrm{K}_0(B) \text{ and } \mathrm{K}_1(A) \simeq \mathrm{K}_1(B).$$

PROPOSITION 4.5 *The K-theory groups are stable under Morita equivalent. If A and B are two $C^\star$-algebras with Morita equivalent underlying rings:*

$$\mathrm{K}_0(A) \simeq \mathrm{K}_0(B) \text{ and } \mathrm{K}_1(A) \simeq \mathrm{K}_1(B).$$

From a split exact sequence of unital $C^\star$-algebras

$$0 \to J \to A \xrightarrow{\pi} A/J \to 0$$

it is possible to define maps $\partial_0 : \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ and $\partial_1 : \mathrm{K}_0(A/J) \to \mathrm{K}_1(J)$ known as *boundary maps* [HR01]. For that, we use the description of K_1 with unitary matrices as in (4). If u is a unitary matrix over A/J, we can lift it up to $a \in A$ such that $\|a\| = 1$. The operators $1 - a^\star a$ and $1 - aa^\star$ define projections over J and:

$$\partial_0([u]) = [1 - a^\star a] - [1 - aa^\star] \in \mathrm{K}_0(J). \tag{5}$$

In the case where $A = \mathscr{B}(H)$ and $J = \mathscr{K}(H)$ are boundary and compact operators over a Hilbert space, the boundary map ∂_0 corresponds to the **Fredholm index** of a. Now, if $[p] - [q] \in \mathrm{K}_0(A/J)$ is a formal difference of two projections p and q over A/J, we can lift them to self-adjoint matrices P and Q over A. The matrices $\exp(2i\pi P)$ and $\exp(2i\pi Q)$ are then unitary over the $C^\star$-algebra J and we get:

$$\partial_1([p] - [q]) = [\exp(2i\pi P)] - [\exp(2i\pi Q)] \in \mathrm{K}_1(J). \tag{6}$$

This boundary map ∂_1 is often call **exponential map**. These construction are related via the following statement, known as **excision theorem**.

PROPOSITION 4.6 *If $0 \to J \to A \to A/J \to 0$ is an exact sequence of $C^\star$-algebras, then the following is exact:*

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r] & \mathrm{K}_0(A) \arrow[r] & \mathrm{K}_0(A/J) \arrow[d, "\partial_1"] \\
\mathrm{K}_1(A/J) \arrow[u, "\partial_0"] & \mathrm{K}_1(A) \arrow[l] & \mathrm{K}_1(J) \arrow[l]
\end{tikzcd}$$

The fact that K-theory groups preserve homotopy equivalences, Morita equivalences and possess the excision property makes naturally K-theory as a $\mathbb{Z}/2\mathbb{Z}$-graded theory. It cannot be computable using resolutions in this context but the Chern character relates these invariants to De Rham cohomology or periodic cyclic homology and makes it more affordable. It is the purpose of the second chapter of this thesis.

4.3 Traces

Another fundamental tool in K-theory is its compatibility with traces. Indeed, any **trace** on a given $C^\star$-algebra A, which is a linear map $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$, can be extended to a trace on the space of idempotent matrices over A by computing it on the diagonal:

$$\tau(p) := \sum_{i=1}^{n} \tau(p_{ii}).$$

Then any trace τ on A defines a trace $\tau_\star$ on $\mathrm{K}_0(A)$ with $\tau_\star([p] - [q]) := \tau(p) - \tau(q)$.

The first example which is relevant to study is the classical matrix trace Tr. Its K-theoretic version is an isomorphism of abelian groups:

$$Tr_\star : \mathrm{K}_0(\mathscr{M}_n(\mathbb{C})) \xrightarrow{\sim} \mathrm{K}_0(\mathbb{C}) \simeq \mathbb{Z}.$$

This can also be viewed as a consequence of the Morita invariance of the K-groups as $\mathscr{M}_n(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$ for all n.

When $A = \mathscr{C}(X)$ is the space of continuous functions on a compact space X the integration against a fixed measure $\tau(f) := \int_X f$ is interesting to study. Its K-theoretical version is defined on complex vector bundles over X by the formula

$$\tau_\star([E]) = \int_X \dim(E_x) d\nu(x).$$

The computation of the number $\tau_\star(\mathrm{Ind}(D)) \in \mathbb{C}$ where $\mathrm{Ind}(D) \in \mathrm{K}_0(X)$ is the analytic index of an elliptic operator D over X is the aim of the Atiyah-Stinger theorem [Ati70]. It appears to be deeply related to characteristic classes of X and relates the analytic and topological side of the story. This theorem is one of the main motivations for the development of K-theory.

Finally, the **canonical trace** over the reduced $C^\star$-algebra $C_r^\star(\Gamma)$ associated to a discrete group Γ is defined on regular elements by:

$$\tau\left(\sum_{g\in\Gamma} \alpha_g g\right) := \alpha_{e_\Gamma} \in \mathbb{C}. \tag{7}$$

One may wonder in which circumstances the image of $\tau_\star : \mathrm{K}_0(C_r^\star(\Gamma)) \to \mathbb{C}$ lies in $\mathbb{Z}$, which is the *integrity* of the trace. When the group is torsion free, this may prove the Kadison-Kaplansky conjecture (see 6.2) and relies heavily on the surjectivity of the Baum-Connes assembly map (11). For any locally compact group G, traces on $\mathscr{C}_c(G)$ that are relevant to study are orbital integrals and we dedicate a section for these (see 6.3).

The behavior of K_1 with respect to traces is directly related to the Chern character. Indeed, the reason why K_0 behaves well with traces is because the *even* Chern class is a trace over the algebra, while on the other hand, the *odd* Chern class belongs to the odd periodic cyclic homology and is far more complicated to study.

4.4 Examples

- $A = \mathbb{C}$: $\mathrm{K}_0(\mathbb{C})$ is the Grothendieck group of the space of idempotent matrices over $\mathbb{C}$. We can associate to any idempotent matrix its rank, which is an integer, and extend it up to the following map which is an isomorphism:

$$\begin{array}{ccc} \mathrm{K}_0(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{Z} \\ [p]-[q] & \longmapsto & \mathrm{rk}(p)-\mathrm{rk}(q) \end{array}$$

  Also, as the spaces $U_n(\mathbb{C})$ are connected, every unitary matrix define the same class in $\mathrm{K}_1(\mathbb{C}) = 0$.

- $A = \mathscr{K}(H)$ is the space of compact operators over a infinite dimensional Hilbert space: The algebra $\mathscr{K}(H)$ is Morita equivalent to $\mathbb{C}$, and then possesses the same K-theory groups $\mathrm{K}_0(\mathscr{K}(H)) \simeq \mathbb{C}$ and $\mathrm{K}_1(\mathscr{K}(H)) = 0$.

- $A = \mathscr{T}$ is the Toeplitz algebra: This algebra fits into the long exact sequence

$$0 \longrightarrow \mathscr{K}(\ell^2(\mathbb{N})) \longrightarrow \mathscr{T} \longrightarrow \mathscr{C}(S^1) \longrightarrow 0.$$

We computed before the K-theory groups of $\mathcal{K}(H)$ and we can check that $\mathrm{K}_i(\mathscr{C}(S^1)) \simeq \mathrm{K}^i(S^1) \simeq \mathbb{Z}^2$ for $i = 0, 1$. Then, by the excision theorem we get that $\mathrm{K}_0(\mathscr{T}) = \mathbb{Z}$ and $\mathrm{K}_1(\mathscr{T}) = 0$.

- $A = C_r^\star(G)$ where G is a locally compact group: The computation of the K-groups of $C_r^\star(G)$ is one of the key motivation for K-theory and more precisely for the Baum-Connes conjecture [PV81]. Some answers have been provided from different frameworks and for different cases.
  - When G is trivial, $C_r^\star(G) \simeq \mathbb{C}$ and we know $\mathrm{K}_0(\mathbb{C}) \simeq \mathbb{Z}$ and $\mathrm{K}_1(\mathbb{C}) = 0$;
  - When $G = \mathbb{Z}/2\mathbb{Z}$, the reduced $C^\star$-algebra $C_r^\star(\mathbb{Z}/2\mathbb{Z})$ is nothing else that the space of vanishing at infinity function over the Pontryagin duals of $\mathbb{Z}/2\mathbb{Z}$, which is $\mathbb{Z}/2\mathbb{Z}$ itself: $C_r^\star(\mathbb{Z}/2\mathbb{Z}) \simeq \mathscr{C}_0(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{C} \oplus \mathbb{C}$. The K-theory of such a $C^\star$-algebra is $\mathrm{K}_0(C_r^\star(\mathbb{Z}/2\mathbb{Z})) \simeq \mathrm{K}_0(\mathbb{C}) \oplus \mathrm{K}_0(\mathbb{C}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, while $\mathrm{K}_1(C_r^\star(\mathbb{Z}/2\mathbb{Z})) = 0$.
  - More generally when G is finite, we have $\mathrm{K}_0(C_r^\star(G)) \simeq \mathbb{Z}^m$ and $\mathrm{K}_1(C_r^\star(G)) = 0$, where m denotes the number of equivalent classes of unitary irreducible representations of G;
  - When G is abelian, we saw that $C_r^\star(G)$ is isomorphic to $\mathscr{C}_0(\widehat{G})$ via Fourier transform, which induces the descriptions $\mathrm{K}_0(C^\star(G)) \simeq \mathrm{K}^0(\widehat{G})$ and $\mathrm{K}_1(C^\star(G)) \simeq \mathrm{K}^1(\widehat{G})$;
  - When G is compact, the Peter-Weyl theorem states that we can decompose the reduced $C^\star$-algebra of the group as a direct sum of matrix spaces [PW27]:

    $$C_r^\star(G) \simeq \bigoplus_{[\pi] \in \widehat{G}} \mathscr{M}_{d_\pi}(\mathbb{C})$$

    where d_π is the dimension of the representation π. As each summand $\mathscr{M}_{d_\pi}(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$ whose K-groups are $\mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$ and $\mathrm{K}_1(\mathbb{C}) = 0$, we get:

    $$\mathrm{K}_0(C_r^\star(G)) \simeq \sum_{[\pi] \in \widehat{G}} \mathbb{Z} \simeq R(G) \text{ and } \mathrm{K}_1(C_r^\star(G)) = 0. \tag{8}$$

While operator K-theory is well-adapted to $C^\star$-algebras and captures many topological invariants, smooth crossed product algebras and Fréchet completions require homological tools such as periodic cyclic homology. These invariants, linked by the Chern character, enable the study of noncommutative spaces that do not admit a $C^\star$-structure, generalizing the classical de Rham and Hodge theories to the operator-algebraic setting. This will be the purpose of the second chapter below.

5 The Baum-Connes conjecture

A recurrent theme of this chapter has been the importance of reduced group $C^\star$-algebras and the difficulty to compute their K-theory. Even in relatively simple situations, such as non-abelian discrete groups, the structure of $K_i(C_r^\star(G))$ remains highly elusive. By contrast, when groups act on geometric spaces, topological tools often provide computable invariants, such as equivariant K-theory and K-homology.

The assembly map was introduced precisely to bridge this gap. Its guiding philosophy is that the analytic K-theory of a group should not be approached directly, but rather *assembled* from the topological data of proper G-actions. The natural receptacle for such data is the equivariant K-homology of the universal space $\underline{E}G$, which encodes all proper actions of G. The assembly map thus provides a canonical transformation

$$\mu : K_i^G(\underline{E}G) \longrightarrow K_i(C_r^\star(G)),$$

sending geometric cycles — such as equivariant elliptic operators — to analytic classes in the group $C^\star$-algebra. From this perspective, the assembly map generalizes the index theorem: it transforms geometric information into analytic invariants.

In this way, the assembly map serves as a conceptual and computational tool: it is the mechanism that translates accessible topological information into the deep and often inaccessible analytic invariants of noncommutative geometry. The celebrated Baum–Connes conjecture, formulated in the 1980s, asserts that this translation is in fact exact: the assembly map is an isomorphism for all second countable, locally compact groups.

5.1 Equivariant K-theory

We saw that K-theory provides a great geometric tools via the study of vector bundles. They provide information on the base space but also on the representations of the structural group. Indeed, a finite dimensional representation of compact group is nothing else that an equivariant vector bundle over the point. This remark motivates the study of equivariant K-theory which defines a version of classical K-theory where the structural group is fixed. This subsection is deeply inspired by the great introduction [Seg68]. We still take G as a locally compact group.

If X is a topological space with a continuous action of G, a **G-vector bundle** over X is a complex vector bundle $p : E \to X$ such that the total space E is endowed with a continuous action of G, the map p is equivariant, and the translation on the fibers $g : E_x \to E_{gx}$ is an isomorphism of complex vector spaces for all $g \in G$.

The space of G-vector bundle $VB_G(X)$ over the space X modulo equivalences defines a semi-group for the direct sum. We call **G-equivariant K-theory** of the space X, the Grothendieck group associated to it:

$$\mathrm{K}_G^0(X) := \mathrm{Gr}(VB_G(X)).$$

It is an abelian group generated by the formal differences $[E]-[F]$ of G-vector bundles over X. As for K-theory, it defines a contravariant function from the homotopic category of compact spaces to the category of abelian groups. We define **odd equivariant K-theory** to be:

$$\mathrm{K}_G^1(X) := K_G^0(\Sigma X).$$

When the group is trivial, we recover the classical K-theory $K^i(X)$ we defined in §4.

One of the most important statement for equivariant K-theory is the Green-Julg theorem [EM09] which asserts that we can express equivariant K-theory in terms of crossed product algebras. The main statement happens in Kasparov KK-theory but can be expressed in weaker version as follows.

THEOREM 5.1 ***(Green-Julg)*** *When $(\mathscr{C}(X), G, \alpha)$ is a $C^\star$-dynamical system with X and G compact, we can compute equivariant K-theory using crossed product algebra:*

$$K^i_G(X) \simeq K_i(\mathscr{C}(X) \rtimes_\alpha G).$$

With this theorem, one has several interesting examples of computation due to the behavior of crossed product algebras. When G is compact, we already know that $\mathrm{K}_0(C^\star(G))$ is isomorphic to its representation ring $R(G)$. We also know that $C^\star(G)$ is nothing other that the crossed product algebra associated to the action of G on the trivial space. Then

$$K^0_G(pt) \simeq K_0(\mathscr{C}(pt) \rtimes_{id} G) \simeq K_0(C^\star(G)) \simeq R(G).$$

Now, when H is a closed subgroup of locally compact group G, one can always define a class in the equivariant K-theory of the homogeneous space G/H from a representation of H:

$$(\pi, V) \in R(H) \longmapsto [G \times_H V \twoheadrightarrow G/H] \in K^0_G(G/H) \tag{9}$$

When G is a compact group, this construction is an isomorphism. Indeed, the formula (3) informs us that crossed product algebra associated to the translation of G on G/H is Morita equivalent to the representation ring $C^\star(H)$. As K-theory is Morita invariant (4.5) and due to the Peter-Weyl theorem (8) we obtain the isomorphism:

$$K^0_G(G/H) \simeq K_0(\mathscr{C}(G/H) \rtimes_\alpha G) \simeq K_0(C^\star(H)) \simeq R(H).$$

Finally, when a compact group G acts freely and properly on a compact space X, due to (3.1) we can compute the equivariant K-theory in terms of the orbit space:

$$K^i_G(X) \simeq K_i(\mathscr{C}(X) \rtimes_\alpha G) \simeq K_i(\mathscr{C}(X/G)) \simeq K^i(X/G),$$

where the isomorphism from left to right is given by the K-theoretic image of the projection $X \to X/G$.

The equivariant theory provides a tool to compute homological invariants of geometric crossed product algebras. It tends to understand spaces of orbits, lying between representation theory and classical geometry.

5.2 Equivariant K-homology

Let X be a locally compact space, endowed with a proper action α of a countable locally compact group G. This section proposes an analytic model for equivariant K-homology which is deeply inspired from the great references [Val02] and [HR01].

If π is a representation of $\mathscr{C}_0(X)$ on a Hilbert space H, we say that a bounded operator T on H is **properly π-supported** when for every $f \in \mathscr{C}_c(X)$ it exists $g \in \mathscr{C}_c(Y)$ such that $T\pi(f) = \pi(g)T\pi(f)$. A **generalized G-elliptic operator** over X is the data of:

- a covariant representation (U,π) of the dynamical system $(\mathscr{C}_0(X), G, \alpha)$ (see 3.2) on some Hilbert space H;
- a self-adjoint properly π-supported operator F on H which is G equivariant, *i.e.* $FU_g = U_g F$ for all $g \in G$, and such that the operators

$$\pi(f)(F^2-1) \text{ and } [\pi(f), F]$$

  are compact for all $f \in \mathscr{C}_0(X)$.

When X is compact, the compactness conditions on the operator F are equivalent to the *Fredholmness* of the operator itself.

A generalized G-elliptic operator (U,π,F) is **even** if the underlying $H = H_0 \oplus H_1$ is $\mathbb{Z}/2\mathbb{Z}$-graded and U, π preserve the graduation while F reverses it:

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix},\ \pi = \begin{pmatrix} \pi_0 & 0 \\ 0 & \pi_1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & P^\star \\ P & 0 \end{pmatrix}.$$

The generalized G-elliptic operator (U,π,F) is **odd** otherwise. We say that the cycle is **degenerate** if for all $f \in \mathscr{C}_0(X)$:

$$\pi(f)(F^2-1) = 0 \text{ and } [\pi(f), F] = 0.$$

Here are some examples:

1. $X = \mathbb{R}$ and $G = \mathbb{Z}$ for the integer translation action: We can equip the Hilbert space $H = \mathscr{L}^2(\mathbb{R})$ with the covariant representation (U,π) where U is the induced translation representation by $\mathbb{Z}$ and π the pointwise multiplication by $\mathscr{C}_0(\mathbb{R})$. The Hilbert transform F is a self-adjoint properly π-supported operator on $\mathscr{L}^2(\mathbb{R})$. One can check that (U,π,F) defines an odd $\mathbb{Z}$-cycle on $\mathbb{R}$.
2. $X = S^1$ and $G = \{1\}$: The Hilbert space $H = \mathscr{L}^2(S^1)$ is naturally equipped with the covariant representation $(U = id, \pi)$ where π is the pointwise multiplication by $\mathscr{C}_0(S^1)$. We define the operator F on the trigonometric basis $(e^{2i\pi n\theta})_{n\in\mathbb{Z}}$ to be $F = \mathrm{diag}(\mathrm{sign}(n))_{n\in\mathbb{Z}}$. It corresponds exactly to the signature of the differential operator $D = -i\frac{d}{d\theta}$. One can check that (id,π,F) is an odd cycle of S^1.

Two cycles $c_0 = (U_0,\pi_0,F_0)$ and $c_1 = (U_1,\pi_1,F_1)$ of same parity are said to be **homotopic** if $U_0 = U_1$, $\pi_0 = \pi_1$ and if there exists a norm continuous path $(F_t)_{t\in[0,1]}$ connecting F_0 to F_1 such that for each $t \in [0,1]$ the triple $c_t = (U_0,\pi_0,F_t)$ is a cycle of same parity again. The cycles c_0 and c_1 are said to be **equivalent** and denoted $c_0 \sim c_1$ if there exists two degenerate cycles d_0 and d_1 such that, up to unitary equivalence, $c_0 \oplus d_0$ is homotopic to $c_1 \oplus d_1$.

We call G**-equivariant K-homology**, and we write $K_0^G(X)$ and $K_1^G(X)$, for the sets of equivalence classes of cycles over X that are *even* and *odd* respectively. These are *abelian groups*. When $G = \{1\}$ is the trivial group, we denote $K_i(X) := K_i^{\{1\}}(X)$ for the **K-homology groups** of X.

Equivariant K-homology is stable under equivariant homotopy; it is a result of G. Kasparov [Kas81]. Indeed, if $f,g : X \to Y$ are G-homotoopic maps over G-spaces we have an isomorphism of abelian groups:

$$K_i^G(X) \simeq K_i^G(Y) \text{ for } i = 0,1.$$

5.2.1 Dirac fundamental class

When G acts properly on a spin^c-manifold X, we can define a class in $K^G_{\dim(X)}(X)$ called **Dirac fundamental class**. Due to the spin^c-structure of X, we can take $D : \Gamma(S) \to \Gamma(S)$ to be the Dirac operator associated to the spinor bundle $S \to X$. The Hilbert space $H = \mathscr{L}^2(X)$ is equipped with the covariant representation (U,π) where U the induced action of G and π the pointwise multiplication from $\mathscr{C}_0(X)$. Finally, the operator:

$$F = \begin{cases} \frac{D}{\sqrt{1+D^2}} & \text{if } \dim(X) \text{ even} \\ \frac{D}{|D|} & \text{if } \dim(X) \text{ odd} \end{cases}$$

is self-adjoint, properly π-supported and defines the Dirac fundamental class

$$[D_X] = [(U,\pi,F)] \in K^G_{\dim(X)}(X).$$

The example 2. above corresponds to the definition of the Dirac fundamental class in the odd case for $X = S^1$ and $G = \{1\}$.

5.2.2 Kasparov-Poincaré duality

Any G-vector bundle E over the spin^c-manifold X induces the Dirac bundle $E \otimes S$ over X via the tensor product with the spinor bundle S over X. The associated Dirac operator is denoted D_E and defines the **twisted Dirac fundamental class** $[D_E] \in K^G_{\dim(X)}(X)$ with respect to E. We can do the same process for the odd case using the definition of higher K-groups via suspension. This construction stands as a cap product in K-homology and generalizes the role of the fundamental class in the De Rham theory [EM07].

THEOREM 5.2 (*Kasparov-Poincaré duality*) *When the countable locally compact group G acts on a spin^c-manifold X, we have the following isomorphism of abelian groups:*

$$\begin{array}{rccc} \mathrm{KPD}: & K^i_G(X) & \xrightarrow{\sim} & K^G_{i+\dim(X)}(X) \\ & [E] & \longmapsto & [D_E] \end{array}$$

Combining the Kasparov-Poincaré duality and the description equivariant K-theory by the Green-Julg theorem, we obtain the following. When a locally compact group G acts freely and properly on a locally compact space X we have the following isomorphism of abelian groups:

$$K_i^G(X) \xrightarrow{\sim} K_i(X/G),$$

where the map is given by the projection $X \to X/G$. Through this isomorphism, the two odd cycles in the examples 1. and 2. define the same class in $K_1^{\mathbb{Z}}(\mathbb{R}) \simeq K_1(S^1)$.

5.3 The assembly map and the Baum-Connes conjecture

The geometric side of the Baum-Connes assembly map encapsulates the data of proper actions of G. A space $\underline{E}G$ equipped with a proper action of G is said to be **universal** if it is paracompact with a metrizable orbit space $\underline{E}G/G$ and if for every proper metrizable G-space X with X/G paracompact, there exists a unique G-equivariant continuous map $X \to \underline{E}G$ up to equivariant homotopy. Such a universal space is unique up to equivariant homotopy. The orbit space $\underline{E}G/G$ encodes the *geometry of proper actions* of the group.

For instance, when G is discrete and torsion free $\underline{E}G$ is nothing other than the universal covering space EG and EG/G is the classifying space BG. In that case the space $\underline{E}G$ captures geometrically the information coming from the group since

$$K_i^G(\underline{E}G) \simeq K_i(BG),$$

and the singular homology of the classifying space BG computes the integer homology of the group G. Also, when G is finite $\underline{E}G = \{pt\}$ and the Kasparov-Poincaré duality states the isomorphism:

$$K_i^G(\underline{E}G) \overset{\text{KPD}}{\simeq} K_G^i(pt) \simeq R(G).$$

Finally, when G is a connected reductive Lie group and K a maximal compact subgroup, the homogeneous space G/K gives a local model for the universal proper space $\underline{E}G$ and we have:

$$\mathrm{K}_i^G(\underline{E}G) \simeq \mathrm{K}_i^G(G/K) \tag{10}$$

The *assembly map* is a function that sends a class of $\mathrm{K}_i^G(\underline{E}G)$ to a class of $K_i(C_r^\star(G))$, *assembling* a bridge from *geometric* to *analytic* sides [HR01] and [Val02].
Let $[(H,U,\pi,F)] \in K_i^G(\underline{E}G)$ be a class represented by a properly supported operator F. The dense subspace $V := \pi(\mathscr{C}_c(\underline{E}G))H \subseteq H$ carries a natural $\mathscr{C}_c(G)$-valued inner product

$$\langle \xi_1, \xi_2 \rangle (g) = \langle \xi_1, U_g \xi_2 \rangle_H.$$

Then, completing V for this product yields a Hilbert $C_r^\star(G)$-module $\mathscr{V}$. The operator F extends as an adjointable operator $\mathscr{F}$ over $\mathscr{V}$. Depending on the parity of the cycle, we obtain the following two constructions.

- In the even case, as the operator F is taken to be properly supported and equivariant, the extended operator $\mathscr{F}$ is invertible modulo compact $C_r^\star(G)$-linear operators, and then *Fredholm* on $\mathscr{V}$. Its kernel and cokernel are finitely generated modules over $C_r^\star(G)$, hence the definition of the **analytic index** of $\mathscr{F}$:

$$\mathrm{Ind}^0(\mathscr{F}) := [\ker(\mathscr{F})] - [\operatorname{coker}(\mathscr{F})] \in K_0(C_r^\star(G)).$$

- In the odd case, the self-adjoint operator $\mathscr{F}$ is invertible modulo compact operators, so its *exponential* $\exp(i\pi\mathscr{F})$ defines a unitary operator modulo compacts. By standard arguments involving *Calkin algebra* and the six-term exact sequence in K-theory (see [Bla12] and 4.6), it determines a class called **odd analytic index** of $\mathscr{F}$:

$$\mathrm{Ind}^1(\mathscr{F}) := [\exp(i\pi\mathscr{F})] \in K_1(C_r^\star(G)).$$

These classes in both cases are heavily related to the boundary maps (5) and (6) defined above. This construction, known as *Mishchenko–Fomenko index*, depends only on the class in the equivairant K-homology $K_i^G(\underline{E}G)$ and then defines a map called **assembly map** [MF80]:

$$\begin{array}{rccc} \mu^i: & K_i^G(\underline{E}G) & \longrightarrow & K_i(C_r^\star(G)) \\ & [(H,U,\pi,F)] & \longmapsto & \mathrm{Ind}^i(\mathscr{F}) \end{array} \tag{11}$$

CONJECTURE 5.3 (*Baum-Connes*) *The assembly map is an isomorphism.*

As the left hand side tends to be more easily accessible than the right hand side, one usually views the Baum-Connes conjecture as an *"explanation"* of the right hand side. The original formulation of this striking conjecture by A. Connes and P. Baum was done in 1982 without using equivariant K-homology as the notion was not yet common. The conjecture sets up a correspondence between different areas of mathematics as the left-hand side is of geometric nature while the right-hand side is a purely analytical object. Let's draw cases where the assembly map is well understood.

When G is discrete and H is a finite subgroup, $\underline{E}G$ is modeled by G/H. In that case, any finite dimensional irreducible representation $(\rho,V)\in R(H)$ induces a cycle in $K_0^G(G/H)$ whose underlying Hilbert space is the induced space $\mathrm{Ind}_H^G V$ as in (**??**). Its image through even the assembly map $\mu_0 : K_0^G(G/H) \to K_0(C_r^\star(G))$ is the class of the projection p_ρ on $\mathbb{C}[H]$ such that $V = \mathrm{Im}(p_\rho)$, but viewed as a projection on $C_r^\star(G)$. In other words we have:

$$\mu_0\left(\mathrm{Ind}_H^G V\right) = [p_\rho] \in K_0(C_r^\star(G)).$$

Also, when $G=\mathbb{Z}$ the left-hand side is the K-homology of $B\mathbb{Z} = S^1$. The assembly map is an isomorphism obtain in that case as the composition of the Kasparov-Poincaré duality (5.2) and the Pontraygin duality (2):

$$K_i^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \simeq K_i(S^1) \overset{\text{KPD}}{\simeq} K^i(S^1) \simeq K_i(\mathscr{C}(S^1)) \overset{\text{Pon.D}}{\simeq} K_i(C_r^\star(\mathbb{Z})).$$

This conjecture doesn't hold yet but has been proved for the several classes of groups by different methods:

- connected reductive Lie groups: uses Arthur's classification of the unitary dual and deep structural properties of the reduced $C^\star$-algebra of the group (by A. Wassermann in 1987);
- groups with Haagerup property: relies on *E-theory* and KK-theory analytic tools for the surjectivity of the assembly map while the injectivity uses constructions of *Dirac-dual Dirac elements* [HK97];
- groups with Rapid Decay property (RD) and a proper, cocompact, isometric action on a strongly bolic metric space: uses *Banach KK-theory* as an extension of Kasparov's KK-theory, geometric estimates and analytic properties of group actions on strongly bolic metric spaces [Laf98];
- groups that admit a finite presentation with only one relation: applies geometric group theory techniques, the RD property and employs some new analytic tools adapted to one-relator groups [BBV99];

- algebraic groups on characteristic zero local fields: employs tools from p-adic representation theory and equivariant KK-theory, reducing to well-understood cases [CEN03];
- Gromov hyperbolic groups and their subgroups: relies on coarse geometric insights and controlled operator algebra techniques and employs cyclic homology and Chern character methods to relate analytic and geometric K-homology [Pus12];
- other cases due to G. Skandalis, B. Bekka, P. de la Harpe et A. Valette, using harmonic analysis, operator-algebraic methods, and representation theory via orbital integrals and character formulas.

The main example for which the conjecture is still not proved is the discrete group $SL_3(\mathbb{Z})$. The conjecture can also be stated *with coefficients* in a $C^\star$-algebra A equipped with an action of G, using crossed product algebras instead of group $C^\star$-algebra. The left-term of the conjecture uses KK-theory with coefficients: $KK_i^G(\mathscr{C}_0(\underline{E}G), A)$. It is defined as the space of generalized G-elliptic operators with underlying A-module Hilbert space.

CONJECTURE 5.4 (*Baum-Connes with coefficients*) *The following assembly map is an isomorphism:*

$$\mu_A^i : KK_i^G(\mathscr{C}_0(\underline{E}G), A) \longrightarrow \mathrm{K}_i(A \rtimes_{\alpha,r} G).$$

Some counter-examples have been found for this extended conjecture by N. Higson, V. Lafforgue and G. Skandalis on graph groups [HLS02], using the works of M-L. Gromov.

5.4 Connes-Kasparov Theorem

The Connes-Kasparov theorem stands a *Lie-version* of the Baum-Connes assembly map. We fix G to be a connected reductive Lie group and K as a maximal compact subgroup of G. The homogeneous space G/K is homeomorphic to a certain $\mathbb{R}^q$, and then inherits from a spin^c-structure manifold. Also, the Cartan decomposition realizes G as a product of manifolds

$$G = K \times G/K$$

but **not** as a group product. Due to the contractible structure of G/K one may expect that group properties of G and K are related; this is the aim of the Connes-Kasparov conjecture.

From any finite dimensional representation V of the compact group K, we can define a G-equivariant vector bundle $E_V = G \times_K V$ over G/K, as in (9). If we compose this construction with the Kasparov-Poincaré duality, we get a map:

$$\begin{array}{ccccc} R(K) & \longrightarrow & K_G^0(G/K) & \overset{KPD}{\longrightarrow} & K^G_{\dim(G/K)}(G/K) \\ V & \longmapsto & [E_V] & \longmapsto & [D_V] := [D_{E_V}] \end{array}$$

As stated in (10), the symmetric space G/K gives a local description of the universal proper G-space, and thus $K_i^G(\underline{E}G) \simeq K_i^G(G/K)$. Through this identification, the assembly map μ must send the class $[D_V] \in K^G_{\dim(G/K)}(G/K)$ to a class in $K_{\dim(G/K)}(C_r^\star(G))$. This construction from a representation of K to a class in the K-theory of the reduced $C^\star$-algebra of G is called **Dirac induction** and denoted D-Ind_K^G.

CONJECTURE 5.5 (*Connes-Kasparov*) *The Dirac induction is an isomorphism:*

$$\text{D-Ind}_K^G : R(K) \longrightarrow \mathrm{K}_{\dim(G/K)}(C_r^\star(G)).$$

This theorem provides a powerful link between representation theory and operator algebras. For reductive Lie groups, the Connes–Kasparov conjecture was proved in two ways by A. Wassermann in 1987 and V. Lafforgue in 1998 [Laf98]. A third way to prove the Connes-Kasparov conjecture has long been suspected to exist. As A. Connes and N. Higson insisted, this meant that the Connes-Kasparov isomorphism could be the noncommutative geometric counterpart of a representation theoretic phenomenon. The reformulation of the Connes-Kasparov conjecture in terms of deformations reflects the observations by G. W. Mackey and leads naturally to the so-called *Mackey's Analogy in K-theory*, which is the purpose of the section §7.

6 Related conjectures

The Baum–Connes conjecture does not live in isolation: it is part of a wider landscape of deep conjectures linking topology, analysis, and algebra. Among them, the Novikov conjecture, the Kadison–Kaplansky conjecture, and problems involving orbital integrals illustrate different facets of the same central question: how do geometric invariants of groups and spaces manifest in analytic and operator-algebraic frameworks? This section surveys these conjectures, emphasizing their interconnections and their role as motivating problems in noncommutative geometry.

6.1 Novikov conjecture

If M is a $4k$-dimensional manifold, the composition of cup product together with the evaluation at the fundamental class $[M] \in H_{4k}(M,\mathbb{R})$ gives a symmetric bilinear form:

$$H^{2k}(M,\mathbb{R}) \times H^{2k}(M,\mathbb{R}) \xrightarrow{-\cup-} H^{4k}(M,\mathbb{R}) \xrightarrow{\langle -,[M]\rangle} \mathbb{R}$$

whose signature is called **signature of the manifold** and denoted $\mathrm{sgn}(M)$. In the 60's, F. Hirzebruch proved that we can express this signature via a k-variable universal polynomial L_k which is independent on the manifold. Indeed, for all manifolds of dimension $4k$ we have the formula:

$$\mathrm{sgn}(M) = \langle L_k(p_1,\cdots,p_k),[M]\rangle \in \mathbb{R},$$

where the p_i stand for the Pontryagin classes of M. The **Hirzebruch L-class** of M is defined to be the class

$$L(M) := \sum_k L_k(p_1,\cdots,p_k) \in H^\star(M,\mathbb{R}).$$

F. Hirzebruch, R. Thom and L. Pontryagin proved the following striking theorem [Tho54].

THEOREM 6.1 *$\langle L(M), [M]\rangle$ defines an homotopy invariant.*

Now, if Γ stands as the fundamental group $\Gamma = \pi_1(M)$ of the manifold, the classifying map $f : M \to B\Gamma$ of the universal covering $\widetilde{M}$ over M sends any cohomological class $x \in H^\star(\Gamma, \mathbb{Q})$ to a class $f^\star(x) \in H^\star(M, \mathbb{Q})$ and defines a **higher signature of** M:

$$\mathrm{sgn}_x(M) := \langle L(M) \cup f^\star(x), [M]\rangle.$$

CONJECTURE 6.2 (*Novikov*) *Higher signatures are homotopy invariants.*

This conjecture bridges *differential geometry* and *topology of fundamental groups*. Thanks to M. Atiyah, I. Singer, A. Connes, G. Kasparov and others, the conjecture can be reformulated using indices of elliptic operators, group $C^\star$-algebras and K-theory. We define the **signature operator** D_s of the manifold to be the sum of the De Rham operator d and its adjoint $d^\star = - \star d \star$, where $\star$ stand as the Hodge star operator: $D_s := d + d^\star$. It can be extended uniquely to a Γ-equivariant operator $\widetilde{D_s}$ on the universal covering $\widetilde{M}$ and then defines a class $[\widetilde{D_s}] \in K_0^\Gamma(\widetilde{M})$. When the fundamental group Γ is torsion free $K_0(B\Gamma) \simeq K_0^\Gamma(\underline{E}\Gamma)$, and the pull back of $[\widetilde{D_s}]$ via the classifying map f gives a class in $\mathrm{K}_0^\Gamma(\underline{E}G)$ whose image through the assembly map is called **Γ-equivariant index of D_s**:

$$\mathrm{Ind}_\Gamma(D_s) := \mu(f_\star([\widetilde{D_s}])) \in K_0(C_r^\star(\Gamma)).$$

Its main property is that its image through the canonical trace τ of Γ (see (7)) computes the signature of M, as stated in [CM90] and [HR01]:

$$\tau_\star(\mathrm{Ind}_\Gamma(D_s)) = \mathrm{sgn(M)} \in \mathbb{R}.$$

Now, higher signatures can be interpreted as *twisted* equivariant indices of D_s. Any element $x \in H^\star(\Gamma, \mathbb{Q})$ can be viewed defines an element in $HP_0(\mathbb{C}[\Gamma])$ (see second chapter) and the Chern-Connes pairing produces a trace on $K_0(\mathbb{C}[\Gamma])$ which can be extended to:

$$(\tau^x)_\star : K_0(C_r^\star(\Gamma)) \to \mathbb{C}.$$

The image of the Γ-equivariant index of D_s through this trace computes higher signatures:

$$(\tau^x)_\star(\mathrm{Ind}_\Gamma(D_s)) = \mathrm{sgn}_x(M) \in \mathbb{R}.$$

The difficulty to show that higher signatures are homotopy invariant then relies on the injectivity of the assembly map. Thus, the following conjecture implies the Novikov conjecture.

CONJECTURE 6.3 *The assembly map is rationally injective.*

This links between Novikov conjecture and the behavior of the assembly map means that *μ encodes all higher signatures at once*. The proof of this conjecture in the K-theoretical setting for *hyperbolic groups* is due to M. Gromov and Connes-Moscovici and the case of amenable groups has been done by Higson-Kasparov and Guentner-Higson-Trout.

6.2 Kadison-Kaplansky conjecture

CONJECTURE 6.4 (*Kadison-Kaplansky*) *When* Γ *is a torsion free discrete group, the only idempotent of* $C_r^\star(\Gamma)$ *are exactly* 0 *and* 1.

While the Novikov conjecture relies on the rational injectivity of the assembly map, this conjecture relies essentially on its surjectivity. The first step is to show that if the canonical trace τ defined in (7) is integer valued, then the conjecture holds. If e is an idempotent of $C_r^\star(\Gamma)$ we have:

$$\tau(e)+\tau(1-e)=\tau(ee^\star)+\tau((1-e)(1-e^\star))=1.$$

Then if $\mathrm{Im}(\tau_\star)\subseteq\mathbb{Z}$, either $\tau(e)=\tau(ee^\star)=0$ and $e=0$ by faithfulness of the trace, or $\tau(1-e)=0$ and $e=1$. Now, a result due to A. J. Berrick, I. Chatterji and G. Mislin [BMG10] shows that when Γ is a torsion free discrete group:

$$\mathrm{Im}(\tau_\star\circ\mu^\Gamma)\subseteq\mathbb{Z}.$$

If the assembly map is onto, then $\mathrm{Im}(\tau_\star)\subseteq\mathbb{Z}$ and the Kadison-Kaplansky holds: there are no non-trivial idempotents in $C_r^\star(\Gamma)$. A great introduction to the Baum-Connes conjecture via the Kadison-Kaplansky conjecture is given by [Val02].

6.3 Orbital integrals

As we have seen in (7), the canonical trace on the reduced $C^\star$-algebra of a locally compact group G induces a relevant map on $\mathrm{K}_0(C_r^\star(G))$. We can compute the value of this K-theoretical trace for different classes of representations. Indeed, for the discrete series this canonical trace gives the formal degree of the representation while it vanishes for the other representations [CM82]. A natural generalization of this trace involves orbital integrals and then can be used to extract properties from $K_0(C_r^\star(G))$. For a semisimple element $g\in G$, the **orbital integral** $\tau_g(f)$ of a function f on G is defined to be the integral of f over the conjugacy class of g:

$$\tau_g(f):=\int_{G/G_g} f(hgh^{-1})dh,$$

where G_g stands as the centralizer of g in G. We recover the canonical trace with $g=e_G$. These integrals converge for functions in the Harish-Chandra's Schwartz algebra $\mathscr{S}(G)$ whose K-theory is the same as the redued $C^\star$-algebra of the group:

$$\mathrm{K}_i(\mathscr{S}(G))\simeq\mathrm{K}_i(C_r^\star(G)). \tag{12}$$

Then orbital integrals lead to K-theoretic traces:

$$\tau_g:\mathrm{K}_0(C_r^\star(G))\longrightarrow\mathbb{C}.$$

If D is an elliptic operator associated to a G-vector bundle over a space X, one may be interested on the image of its equivariant index trough the integral orbital traces:

$$\tau_g(\mathrm{Ind}_G(D))\in\mathbb{C}.$$

Fixed point formulas have been showed for these numbers when G is discrete [WW13] and of Lie-type [HW17]. For the discrete case, these formulas have consequences on orbifold geometry and positive scalar curvature metrics while the Lie case generalizes a Harish-Chandra's character formula. More recently, an explicit formula have been computed for the $\tau_g(\mathrm{Ind}_G(D))$ [HW18]. It turns out these vanish if $\mathrm{rk}(G) \neq \mathrm{rk}(K)$ but has interesting consequences if their rank are the same. For instance, it spotlights an embedding of $K_0(C_r^\star(G))$ in the space of distributions on regular elements of G, an induction formula from K-equivariant indices to G-equivariant indices and a K-theoretical version of the vanishing Selberg principle [BB92].

7 Mackey analogy

The Mackey analogy draws a conceptual bridge between the representation theory of reductive Lie groups and that of simpler "motion groups" associated to their maximal compact subgroups. This analogy, reformulated in operator-algebraic terms, provides insight into the structure of the unitary dual and clarifies the role of crossed product algebras in representation theory. In this part we review the Cartan motion group, explain its relation to equivariant K-theory, and discuss Connes–Higson's deformation picture of the assembly map, which encapsulates the geometric content of the Connes–Kasparov theorem.

REFAIRE LA PARTIE

7.1 The Cartan motion group

The **Cartan motion group** G_0 is defined to be the semi-direct product

$$G_0 = K \rtimes P$$

where the action of K on P is given by $k \cdot X := ad(k)X$. The Lie groups G and G_0 are homeomorphic but don't possess the same group structure. In the 1970's, Mackey suggested that there should be a natural bijection between the tempered dual of G and the unitary dual of G_0, which have been proved years later as the Mackey-Higson bijection:

$$\mathcal{M} : \widehat{G}_0 \longleftrightarrow \widehat{G}_t.$$

Because $G_0 = K \rtimes \mathfrak{p}$ is an extension of K by a vector abelian group of even dimension, G_0 is amenable and $C_r^\star(G_0) = C^\star(G_0)$. Also, the equivariant Bott periodicity theorem yields a natural isomorphism between $K_0(C^\star(G_0))$ and $R(K)$. Under this identification, the Connes-Kasparov conjecture becomes the comparison:

$$K_0(C^\star(G_0)) \longrightarrow K_0(C_r^\star(G)).$$

7.2 Connes-Higson picture for the assembly map

As G_0 and G possess the same manifold structure, it is natural to think on a *continuous* family of groups $(G_t)_{t\in\mathbb{R}}$ which interpolates between $G_1 = G$ and G_0, these are called

deformation groups. For every non-zero t, we define G_t to be the set G equipped with the topology that makes the following map an homeomorphism:

$$\begin{array}{cccc} \phi_t: & K\times P & \longrightarrow & G \\ & (k,X) & \longmapsto & k\exp_G(tX) \end{array}$$

The disjoint union $\mathscr{G} = \sqcup_{t\in\mathbb{R}} G_t$ is a topological space with the topology inherited from

$$\begin{array}{ccc} \mathscr{G} & \longrightarrow & G_0 \cup (G\times\mathbb{R}^\times) \\ g\in G_t & \longmapsto & \begin{cases} g & \text{if } t=0 \\ (\phi_t(g),t) & \text{else} \end{cases} \end{array}$$

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This topology on $\mathscr{G}$ makes the field $\{C_r^\star(G_t)\}_{t\in\mathbb{R}}$ a continuous field of $C^\star$-algebras. We finally define:

$$\mathscr{C} := C^\star\Big(\Gamma\left(\{C_r^\star(G_t)\}_{t\in\mathbb{R}} \to [0,1]\right)\Big).$$

This topology on $\mathscr{G}$ makes the field $\{C_r^\star(G_t)\}_{t\in\mathbb{R}}$ a continuous field of $C^\star$-algebras. We finally define:

$$\mathscr{C} := C^\star\Big(\Gamma\left(\{C_r^\star(G_t)\}_{t\in\mathbb{R}} \to [0,1]\right)\Big).$$

This $C^\star$-algebra encapsulates the whole deformation from the reduced $C^\star$-algebra of G_0 to the reduced $C^\star$-algebra of $G = G_1$. It is naturally equipped with two canonical maps:

$$C^\star(G_0) \xleftarrow{\alpha_0} \mathscr{C} \xrightarrow{\alpha_1} C_r^\star(G).$$

When both of this map are quasi-isomorphisms in K-theory the Connes-Kasparov conjecture holds. As the topology of $\mathscr{G}$ suggests it, it is easier to show that α_0 is a quasi-isomorphism in K-theory, and it is even feasible to find an inverse of it. The tougher proof for the other map α_1 have been proposed by A. Connes and N. Higson in 1990 and recently reformulated by A. Afgoustidis in 2019.

THEOREM 7.1 (*Connes-Higson*) *The map $\alpha_1 \circ \alpha_0^{-1}$ defines a quasi-isomorphism:*

$$K_0(C^\star(G_0)) \xrightarrow{\sim} K_0(C_r^\star(G)).$$

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